A BINOMIAL DIOPHANTINE EQUATION

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1. Introduction

In [3, Section D3], Richard K. Guy asks for the existence of nontrivial solutions of the diophantine equation $\binom{n}{2} = \binom{m}{4}$, other than $\binom{21}{2} = \binom{10}{4} = 210$ (note that the trivial solutions are $\binom{2}{2} = \binom{4}{4} = 1$ and $\binom{6}{2} = \binom{6}{4} = 15$). In this note we will answer this question, by proving the following result.

THEOREM 1. *The only* $(n, m) \in \mathbb{Z}^2$ *with* $n \geqslant 2$ *and* $m \geqslant 4$ *satisfying* $\binom{n}{2} = \binom{m}{4}$ *are* $(n, m) = (2, 4)$, $(6, 6)$, *and* $(21, 10)$.

Our binomial diophantine equation represents an elliptic curve, since it can be rewritten as a quartic polynomial being a square. Indeed, on putting $u = 2n - 1$ and $v = 2m - 3$, we see at once that Theorem 1 follows from the following result.

THEOREM 2. *The only* $(u, v) \in \mathbb{Z}^2$ *with* $u \geqslant 0$ *and* $v \geqslant 0$ *satisfying*

$$48u^2 = v^4 - 10v^2 + 57 \tag{1}$$

are $(u, v) = (1, 1)$, $(1, 3)$, $(3, 5)$, $(11, 9)$, *and* $(41, 17)$.

The elliptic curve defined over $\mathbb{Q}$ represented by equation (1) has rank 2. However, we will not make use of "geometric" properties of this curve, but use only the "algebraic" properties of the equation. The "geometric" approach to quartic equations described by Tzanakis [5] will most probably lead to an efficient alternative way of proving our results.

In Section 2 we will show that Theorem 2 is a consequence of the following result.

THEOREM 3.
(i) *The only* $(x, y) \in \mathbb{Z}^2$ *satisfying*

$$x^4 + 16x^3y - 6x^2y^2 + 304xy^3 + 361y^4 = m \tag{2}$$

with $m \in \{1, 4, 9, 36\}$ *are* $(x, y) = \pm(1, 0)$ *with* $m = 1$, *and* $(x, y) = \pm(1, -1)$ *with* $m = 36$.

(ii) *The only* $(x, y) \in \mathbb{Z}^2$ *satisfying*

$$3x^4 + 48x^3y + 302x^2y^2 + 912xy^3 + 1083y^4 = m \tag{3}$$

with $m \in \{3, 12\}$ *are* $(x, y) = \pm(1, 0)$ *and* $\pm(6, -1)$ *with* $m = 3$, *and* $(x, y) = \pm(3, -1)$ *with* $m = 12$.

In Section 3 we will prove Theorem 3 using the method described in [**6**]. We will use the transcendence result of Baker and Wüstholz [**2**], that yields absolute upper bounds for x,y of the size of $10^{2\times 10^{17}}$. In fact, for the solutions of $\binom{n}{2} = \binom{m}{4}$ we obtain the upper bounds $m < 10^{2\times 10^{17}}$ and $n < 10^{4\times 10^{17}}$. Computational diophantine approximation methods are then used to search in a very efficient way for the solutions below these bounds.

It might be possible to find a proof of Theorem 3 that avoids the deep results of transcendence theory completely, following the line of argument of [**4**, Section 4]. This would lead to an 'elementary' solution of $\binom{n}{2} = \binom{m}{4}$, and we leave this as a challenge to the reader.

Note that the equation $\binom{n}{2} = \binom{m}{3}$ defines an elliptic curve also. This equation has been solved by Avanesov [**1**]. We refer to [**3**, Sections B31, D3] for other results and references to papers on binomial diophantine equations.

In May 1995 Á. Pintér informed me that he obtained the same results as we did, by essentially the same method. His paper will appear in Publ. Math. (Debrecen).

2. Proof of Theorem 2

In this section we derive Theorem 2 from Theorem 3. Write equation (1) as

$$48u^2 = (v^2 - 5)^2 + 32,$$

and factorize over $\mathbb{Q}(\sqrt{-2})$. A common divisor of $v^2 - 5 + 4\sqrt{-2}$ and its complex conjugate also divides their difference $8\sqrt{-2}$. Since 3 splits as $3 = (1 + \sqrt{-2})(1 - \sqrt{-2})$ we obtain

$$v^2 - 5 + 4\sqrt{-2} = (-1)^a(\sqrt{-2})^b(1 + \sqrt{-2})^c(1 - \sqrt{-2})^d(A + B\sqrt{-2})^2, \tag{4}$$

where $a,b,c,d \in \{0, 1\}$, and $A,B \in \mathbb{Z}$. Taking norms in (4) we find from

(1) that $2^b 3^{c+d}$ is 3 times a square, hence $b = 0$, and $(c, d) = (1, 0)$ or $(c, d) = (0, 1)$.

The case $(c, d) = (1, 0)$. In this case we derive from equating real and imaginary parts in (4) the following system:

$$v^2 - 5 = (-1)^a(A^2 - 4AB - 2B^2), \tag{5}$$

$$4 = (-1)^a(A^2 + 2AB - 2B^2). \tag{6}$$

Equation (6) implies that A is even, and subsequently that B is even. Equation (6) is solvable modulo 3 only if $a = 0$. So on putting $A = 2C$, $B = 2D$, and adding $\frac{5}{4}$ times (6) to (5), we find

$$v^2 = 9C^2 - 6CD - 18D^2, \tag{7}$$

$$1 = C^2 + 2CD - 2D^2. \tag{8}$$

Equation (7) can be written as

$$(3C - D + v)(3C - D - v) = 19D^2,$$

so we can factor over $\mathbb{Z}$. Let p be a prime dividing $3C - D + v$ and $3C - D - v$. If $p \neq 2$ and $p \neq 19$ then $p \mid 3C - D$ and $p \mid D$, and since by (8) $\gcd(C, D) = 1$, we have $p = 3$. It follows that there are $e,f,g,h \in \{0, 1\}$ and $E,F \in \mathbb{Z}$ with

$$3C - D + v = (-1)^e 2^f 3^g 19^h E^2, \tag{9}$$

$$3C - D - v = (-1)^e 2^f 3^g 19^{1-h} F^2, \tag{10}$$

$$D = \pm 2^f 3^g EF. \tag{11}$$

For symmetry reasons (the signs of C,D,u,v are irrelevant, and E and F may be interchanged) we may assume without loss of generality that $e = h = 0$, and that in (11) the $\pm$ is $+$. Hence from the system (9), (10), (11) we derive

$$C = 2^{f-1}3^{g-1}(E^2 + 2EF + 19F^2),$$

and substituting this and (11) into (8) we obtain equation (2) with $x = E$, $y = F$, and $m = 2^{2-2f}3^{2-2g} \in \{1, 4, 9, 36\}$.

The case $(c, d) = (0, 1)$. In this case we derive from equating real and imaginary parts in (4) the following system:

$$v^2 - 5 = (-1)^a(A^2 + 4AB - 2B^2), \tag{12}$$

$$4 = (-1)^a(-A^2 + 2AB + 2B^2). \tag{13}$$

Equation (13) implies that A is even, and subsequently that B is even.

Equation (13) is solvable modulo 3 only if $a = 1$. So on putting $A = 2C$, $B = 2D$, and adding $\frac{3}{4}$ times (13) to (12), we find

$$v^2 = C^2 - 26CD - 2D^2, \tag{14}$$

$$1 = C^2 - 2CD - 2D^2. \tag{15}$$

Equation (14) can be written as

$$(C - 13D + v)(C - 13D - v) = 3^2 19 D^2,$$

so we can factor over $\mathbb{Z}$. Let p be a prime dividing $C - 13D + v$ and $C - 13D - v$. If $p \neq 2$, $p \neq 3$ and $p \neq 19$, then $p \mid C - 13D$ and $p \mid D$, which is impossible, since (15) implies $\gcd(C, D) = 1$. It follows that there are $e,f,g,h \in \{0, 1\}$ and $E,F \in \mathbb{Z}$ with

$$C - 13D + v = (-1)^e 2^f 3^g 19^h E^2, \tag{16}$$

$$C - 13D - v = (-1)^e 2^f 3^g 19^{1-h} F^2, \tag{17}$$

$$D = \pm 2^f 3^{g-1} EF. \tag{18}$$

For symmetry reasons (the signs of C,D,u,v are irrelevant, and E and F may be interchanged) we may assume without loss of generality that $e = h = 0$, and that in (18) the $\pm$ is $+$. Hence from the system (16), (17), (18) we derive

$$C = 2^{f-1} 3^{g-1}(3E^2 + 26EF + 57F^2),$$

and substituting this and (18) into (15) we obtain equation (3) with $x = E$, $y = F$, and $m = 2^{2-2f} 3^{1-2g}$. By $m \in \mathbb{Z}$ it follows that $g = 0$, and we have $m \in \{3, 12\}$.

Now clearly Theorem 3 implies Theorem 2.

3. Proof of Theorem 3

We start with studying quartic fields. Let θ_1, θ_2 be roots of

$$\theta_1^4 + 2\theta_1^2 - 2 = 0, \qquad \theta_2^4 - 2\theta_2^2 - 2 = 0.$$

Put

$$\psi_1 = -7 + 8\theta_1 - 3\theta_1^2 + 2\theta_1^3, \qquad \psi_2 = (-11 - 8\theta_2 - \theta_2^2 + 2\theta_2^3)/3,$$

then

$$\psi_1^4 + 16\psi_1^3 - 6\psi_1^2 + 304\psi_1 + 361 = 0,$$

$$3\psi_2^4 + 48\psi_2^3 + 302\psi_2^2 + 912\psi_2 + 1083 = 0.$$

Let $\mathbb{K}_i = \mathbb{Q}(\theta_i)$ for $i = 1,2$. Both these fields are half-real, have discriminant $-4608=-2^9 3^2$, integral basis $\{1, \theta_i, \theta_i^2, \theta_i^3\}$, trivial class group, and Galois group $\mathbb{D}_8$. Nevertheless they are not isomorphic. Fundamental units of $\mathbb{K}_i$ are ϵ_i, η_i, given by

$$\epsilon_1 = 1-\theta_1^2, \qquad \eta_1 = 1+\theta_1,$$
$$\epsilon_2 = 1+\theta_2^2, \qquad \eta_2 = 1-\theta_2-\theta_2^2.$$

The prime ideal decompositions of the primes 2 and 3 in the fields $\mathbb{K}_i$ are as follows:

$$\text{in } \mathbb{K}_1\text{: } (2)=(\theta_1)^4, \qquad (3)=(1+\theta_1^2)^2,$$
$$\text{in } \mathbb{K}_2\text{: } (2)=(\theta_2)^4, \qquad (3)=(1+\theta_2)^2(1-\theta_2)^2.$$

Complete sets of non-associated integral elements $\mu_1 \in \mathbb{K}_1$ of norm $m \in \{1, 4, 9, 36\}$ *are given by*

$$\mu_1 = 1 \quad \text{if} \quad m=1, \qquad \mu_1 = 1+\theta_1^2 \quad \text{if} \quad m=9,$$
$$\mu_1 = \theta_1^2 \quad \text{if} \quad m=4, \qquad \mu_1 = 2-\theta_1^2 \quad \text{if} \quad m=36.$$

Note that the denominator of ψ_2 is $1-\theta_2$. Complete sets of non-associated elements $\mu_2 \in \mathbb{K}_2$ of norm $\frac{1}{3}m \in \{1, 4\}$ and with denominator 1 or $1-\theta_2$ are given by

$$\mu_2 = 1 \quad \text{or} \quad \mu_2 = \frac{1+\theta_2}{1-\theta_2} \quad \text{if} \quad m=3,$$

$$\mu_2 = \theta_2^2 \quad \text{or} \quad \mu_2 = \frac{1+\theta_2}{1-\theta_2}\theta_2^2 \quad \text{if} \quad m=12.$$

Put $\beta_i = x - y\psi_i$ for $i = 1,2$. Now it follows that the Thue equations (2) and (3) are equivalent to

$$\beta_i = \pm \mu_i \epsilon_i^k \eta_i^l \tag{19}$$

for unknown $k, l \in \mathbb{Z}$, with ψ_i, μ_i, ϵ_i, η_i as described above, for $i \in \{1, 2\}$.

We denote the real conjugates of some $\alpha \in \mathbb{K}_i$ by $\alpha^{(1)}$, $\alpha^{(2)}$, and the non-real conjugates by $\alpha^{(3)}$, $\alpha^{(4)}$. We take the conjugates so that $\theta_i^{(1)} > 0$, and $\operatorname{Im} \theta_i^{(3)} > 0$. For convenience we will often suppress the lower index i.

Let $j, j' \in \{1, 2\}$ be such that

$$|\beta^{(j)}| < |\beta^{(j')}|.$$

The Siegel identity for the jth, 3rd and 4th conjugates is

$$(\psi^{(j)}-\psi^{(4)})\beta^{(3)} - (\psi^{(j)}-\psi^{(3)})\beta^{(4)} = (\psi^{(3)}-\psi^{(4)})\beta^{(j)}.$$

Dividing the second term and substituting the 3rd and 4th conjugates of (19), we obtain

$$\frac{\psi^{(j)}-\psi^{(4)}}{\psi^{(j)}-\psi^{(3)}}\frac{\mu^{(3)}}{\mu^{(4)}}\left(\frac{\epsilon^{(3)}}{\epsilon^{(4)}}\right)^k\left(\frac{\eta^{(3)}}{\eta^{(4)}}\right)^l-1=\frac{\psi^{(3)}-\psi^{(4)}}{\psi^{(j)}-\psi^{(3)}}\frac{\beta^{(j)}}{\beta^{(4)}}. \tag{20}$$

Note that $\epsilon^{(3)}=\epsilon^{(4)}$ for both $i=1,2$, that $\mu_1^{(3)}=\mu_1^{(4)}$ for all possibilities for μ_1, that for $\dfrac{\mu_2^{(3)}}{\mu_2^{(4)}}$ there are only two possibilities: 1 or $\left(\dfrac{1+\theta_2^{(3)}}{1-\theta_2^{(3)}}\right)^2$. Further note that all the quotients in the left hand side of (20) are quotients of complex conjugates. So put

$$\rho_j=\frac{1}{\sqrt{-1}}\,\mathrm{Log}\,\frac{\psi^{(j)}-\psi^{(4)}}{\psi^{(j)}-\psi^{(3)}}\frac{\mu^{(3)}}{\mu^{(4)}},\qquad \tau=\frac{1}{\sqrt{-1}}\,\mathrm{Log}\,\frac{\eta^{(3)}}{\eta^{(4)}}.$$

Here we take the principal value of the logarithm, so that $\rho_j,\ \tau\in -\pi,\pi]$. We give some idea of the numerical values:

if $i=1$ then $\tau=2.05341\ldots$, $\rho_1=-2.13977\ldots,\rho_2=-0.44585\ldots$,

if $i=2$ then $\tau=-0.91764\ldots$,

if $\dfrac{\mu_2^{(3)}}{\mu_2^{(4)}}=1$ then $\rho_1=1.62687\ldots$, $\rho_2=-2.96239\ldots$,

if $\dfrac{\mu_2^{(3)}}{\mu_2^{(4)}}=\left(\dfrac{1+\theta_2^{(3)}}{1-\theta_2^{(3)}}\right)^2$ then $\rho_1=-1.82536,\ldots\ \rho_2=-0.13145\ldots.$

Put

$$\Lambda_j=\rho_j+l\tau-z2\pi,$$

where we take $z\in\mathbb{Z}$ such that $\Lambda_j\in(-\pi,\pi]$. Now the left hand side of (20) can be written as $e^{\sqrt{-1}\Lambda_j}-1$. Note that $|z|\leqslant 1+\frac{1}{2}|l|$, so that $\max\{|l|,|z|\}=|l|$ unless $l=0$. In the sequel we will assume that $l\neq 0$ and that $|y|\geqslant 1$.

The theory of linear forms in logarithms of algebraic numbers tells us that Λ_j cannot be near to zero. In fact, the sharp result of Baker and Wüstholz [2] yields (unless $l=0$)

$$|\Lambda_j|>\exp(-C_0\log|l|) \tag{21}$$

for a positive constant C_0 to be given below.

On the other hand, we can follow the arguments of [TdW], and derive an upper bound for $|\Lambda_j|$, namely

$$|\Lambda_j|<C_1\exp(-C_2|l|) \tag{22}$$

for positive constants C_1,C_2, to be given below. Now, combining (21) and (22) we find an absolute upper bound C_3 for $|l|$.

We give the details of the calculation of these constants in an appendix to this paper. The constants are different in three cases:

$$\text{case I:} \quad i = 1,$$

$$\text{case IIa:} \quad i = 2 \quad \text{and} \quad \frac{\mu_2^{(3)}}{\mu_2^{(4)}} = 1,$$

$$\text{case IIb:} \quad i = 2 \quad \text{and} \quad \frac{\mu_2^{(3)}}{\mu_2^{(4)}} = \left(\frac{1+\theta_2^{(3)}}{1+\theta_2^{(4)}}\right)^2.$$

We found the values given in the following Table.

case	C_0	C_1	C_2	C_3
I	1.61069×10^{16}	10580.360	2.553736	2.52774×10^{17}
IIa	1.90462×10^{16}	49.881773	3.7556315	2.02082×10^{17}
IIb	4.09018×10^{16}	823.56656	3.7556315	4.42507×10^{17}

It's not too hard to show that these upper bounds for $|l|$ lead to the upper bounds $m < 10^{2\times 10^{17}}$ and $n < 10^{4\times 10^{17}}$ for the solutions of $\binom{n}{2} = \binom{m}{4}$. Clearly for attempting a direct search these bounds are way too large. Therefore, to find the solutions below these bounds we will work again with the linear forms Λ_j. We follow the path of [6] further, and reduce the upper bound C_3 for $|l|$ to a much friendlier size, by using computational lattice base reduction techniques. This we will show to be possible within a few seconds of computational time.

Take a large enough positive constant C, of the size of C_3^2. Consider the lattice

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 \\ [C\tau] & [C2\pi] \end{pmatrix} \begin{pmatrix} l \\ -z \end{pmatrix} \,\middle|\, l, z \in \mathbb{Z} \right\},$$

(where $[\cdot]$ means rounding to an integer), and define

$$\lambda_j = [C\rho_j] + l[C\tau] - z[C2\pi].$$

By a variant of the euclidean algorithm it is not difficult to compute the distance d_j between the point

$$P = \begin{pmatrix} 0 \\ -[C\rho_j] \end{pmatrix}$$

and the nearest point in the lattice Γ. If C is taken large enough, then we might expect that

$$d > \sqrt{C_3^2 + \left(2 + \frac{3}{2} C_3\right)^2},$$

and in that case we can make the following estimates. Because

$\binom{l}{\lambda_j}$ is a difference vector of a lattice point and the point P, we find $d^2 \leq l^2 + \lambda^2$. By the definition of λ_j we have $|\lambda_j - C\Lambda_j| \leq 1 + |l| + |z| \leq 2 + \frac{3}{2}C_3$, so that

$$|\Lambda_j| > \frac{1}{C}\left(\sqrt{d^2 - C_3^2} - \left(2 + \frac{3}{2}C_3\right)\right). \tag{23}$$

Once we have found this very explicit numerical lower bound for $|\Lambda_j|$, we can use (22) to give a new upper bound for $|l|$.

Initially we took $C = 10^{37}$. Note that we need the values of ρ_j, τ, 2π to at least 37 decimal places behind the decimal point, and that we have to use multi-precision routines. This yielded:

in case I:	$d_1 > 7.20291 \times 10^{18}$,	$d_2 > 7.45412 \times 10^{18}$,	$\|l\| \leq 20$,
in case IIa:	$d_1 > 8.32804 \times 10^{18}$,	$d_2 > 1.04835 \times 10^{19}$,	$\|l\| \leq 12$,
in case IIb:	$d_1 > 7.36079 \times 10^{18}$,	$d_2 > 9.43324 \times 10^{18}$,	$\|l\| \leq 12$.

Then we took $C = 10^4$ with the new upper bounds as C_3. This yielded:

in case I:	$d_1 > 50.9901$,	$d_2 > 93.9414$,	$\|l\| \leq 6$,
in case IIa:	$d_1 > 79.3095$,	$d_2 > 85.7962$,	$\|l\| \leq 2$,
in case IIb:	$d_1 > 98.0204$,	$d_2 > 85.2877$,	$\|l\| \leq 3$.

Total computation time is to be measured in seconds only on a 486/33 personal computer.

Only a few cases remain to be checked, namely those with $|l| \leq 6$, 2 or 3, and those with $y = 0$. This can be done by hand. There are only the following solutions.

i	μ	k	l	z	j	(x,y)
1	1	0	0	0	1,2	$\pm(1,0)$
	θ^2	none				
	$1+\theta^2$	none				
	$2-\theta^2$	1	−2	−1	1	$\pm(1,-1)$
2	1	0	0	0	1,2	$\pm(1,0)$
	θ^2	none				
	$\frac{1+\theta}{1-\theta}$	−2	−2	0	1	$\pm(6,-1)$
	$\frac{1+\theta}{1-\theta}\theta^2$	−1	0	0	2	$\pm(3,-1)$

This completes the proof of Theorem 3, hence of Theorems 1 and 2.

Appendix

In this Appendix we give details of the calculations of the constants C_0, C_1, C_2, C_3 in the three cases I, IIa, IIb. We start with calculating C_0.

We apply the main result of [2], which reads

$$C_0 = 18(n+1)!\, n^{n+1}(32d)^{n+2} \log(2nd) h'(p_j) h'(\tau) h'(1),$$

with in our case the number of terms in Λ_j being $n = 3$, the field degree being $d = 8$, and for the heights we computed $h'(1) = \frac{1}{8}$, and

in case I: $h'(\rho_1) = h'(\rho_2) = 1.355292\ldots,\quad h'(\tau) = 0.638343\ldots,$

in case IIa: $h'(\rho_1) = h'(\rho_2) = 1.089584\ldots,\quad h'(\tau) = 0.938907\ldots,$

in case IIb: $h'(\rho_1) = h'(\rho_2) = 2.339887\ldots,\quad h'(\tau) = 0.938907\ldots.$

This immediately led to the values for C_0 given in the paper. Note that $h'(L) = \log|l|$.

Then we calculate C_1, C_2. We will estimate the conjugates of β in absolute value from below and above. Using the definition of j, j', the fact that (19) implies

$$\prod_{k=1}^{4} |\beta^{(k)}| = |\mathrm{Norm}_{K/\mathbb{Q}}\mu| = \begin{cases} m \in \{1, 4, 9, 36\} \text{ in case I} \\ \frac{1}{3}m \in \{1, 4\} \text{ in cases IIa, IIb} \end{cases}$$

and using the assumption that $|y| \geq 1$, we have

$$|\beta^{(3)}| = |\beta^{(4)}| \geq |\mathrm{Im}\, \psi^{(3)}|\, |y| > \begin{cases} 4.1915652\, |y| \text{ in case I} \\ 2.6991607\, |y| \text{ in cases IIa, IIb} \end{cases},$$

$$|\beta^{(j')}| > \frac{1}{2}(|\beta^{(j)}| + |\beta^{(j')}|) \geq \frac{1}{2}|\beta^{(j)} - \beta^{(j')}|$$

$$= \frac{1}{2}|\psi^{(1)} - \psi^{(2)}|\, |y| > \begin{cases} 8.0974822\, |y| \text{ in case I} \\ 1.3971884\, |y| \text{ in cases IIa, IIb} \end{cases}$$

$$|\beta^{(j)}| = \frac{\mathrm{Norm}_{K/\mathbb{Q}}\mu}{|\beta^{(j')}|\, |\beta^{(3)}|^2} < \begin{cases} 0.25304633\, |y|^{-3} \text{ in case I} \\ 0.39295925\, |y|^{-3} \text{ in cases IIa, IIb} \end{cases},$$

$$|\beta^{(j')}| \leq |\beta^{(j')} - \beta^{(j)}| + |\beta^{(j)}| \leq (|\psi^{(1)} - \psi^{(2)}|$$

$$+ |\beta\gamma^{(j)}|\, |y|^3))\, |y| < \begin{cases} 16.448011\, |y| \text{ in case I} \\ 3.1873361\, |y| \text{ in cases IIa, IIb} \end{cases}$$

$$|\beta^{(3)}| \leq |\beta^{(3)} - \beta^{(j)}| + |\beta^{(j)}|$$

$$\leq \left(\max_{h=1,2} |\psi^{(3)} - \psi^{(h)}| + |\beta^{(j)}|\, |y|^3\right) |y| < \begin{cases} 19.211988\, |y| \text{ in case I} \\ 4.1074722\, |y| \text{ in cases IIa, IIb} \end{cases},$$

$$|\beta^{(j)}| = \frac{\mathrm{Norm}_{K/\mathbb{Q}}\mu}{|\beta^{(j')}|\, |\beta^{(3)}|^2} > \begin{cases} 0.00016471836\, |y|^{-3} \text{ in case I} \\ 0.018596143\, |y|^{-3} \text{ in cases IIa, IIb} \end{cases}.$$

Consequently, we find from (20) that

$$|e^{\sqrt{-1}\Lambda_j}-1|\leq \max_{h=1,2}\left|\frac{\psi^{(3)}-\psi^{(4)}}{\psi^{(h)}-\psi^{(3)}}\right|\left|\frac{\beta^{(j)}}{\beta^{(3)}}\right|<\begin{cases}0.10590719\,|y|^{-4} \text{ in case I}\\ 0.29000349\,|y|^{-4} \text{ in cases IIa, IIb}\end{cases},$$

which by $|y|\geq 1$ leads to

$$|\Lambda_j|<\begin{cases}0.10595675\,|y|^{-4} \text{ in case I}\\ 0.29102947\,|y|^{-4} \text{ in cases IIa, IIb}\end{cases}. \tag{24}$$

From (19) we derive

$$\begin{pmatrix}k\\ l\end{pmatrix}=\begin{pmatrix}\log|\epsilon^{(1)}| & \log|\eta^{(1)}|\\ \log|\epsilon^{(2)}| & \log|\eta^{(2)}|\end{pmatrix}^{-1}\begin{pmatrix}\log|\beta^{(1)}|-\log|\mu^{(1)}|\\ \log|\beta^{(2)}|-\log|\mu^{(2)}|\end{pmatrix},$$

and because $\epsilon^{(1)}=\epsilon^{(2)}$ we find

$$l=\frac{1}{\log|\eta^{(1)}|-\log|\eta^{(2)}|}\left(\log\left|\frac{\beta^{(1)}}{\beta^{(2)}}\right|-\log\left|\frac{\mu^{(1)}}{\mu^{(2)}}\right|\right),$$

from which we derive

$$|l|\leq\frac{1}{|\log|\eta^{(1)}|-\log|\eta^{(2)}||}\left(\log\left|\frac{\beta^{(j')}}{\beta^{(j)}}\right|+\left|\log\left|\frac{\mu^{(1)}}{\mu^{(2)}}\right|\right|\right).$$

So for our three cases we find

$$|l|<\begin{cases}4.5083406+1.5665549\log|y| \text{ in case I}\\ 1.3696728+1.0650672\log|y| \text{ in case IIa}\\ 2.1162819+1.0650672\log|y| \text{ in case IIb}\end{cases}$$

Combined with (24) this yields the values for C_1 and C_2 given in the paper.

Now combining (21) and (22) we find the given values for C_3.

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